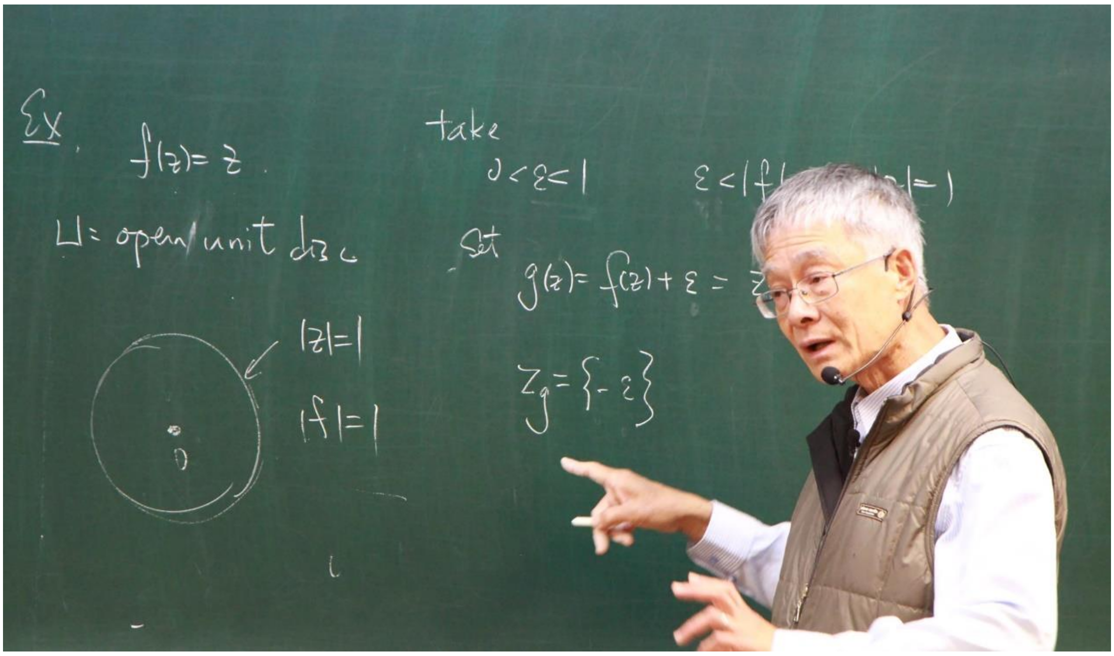
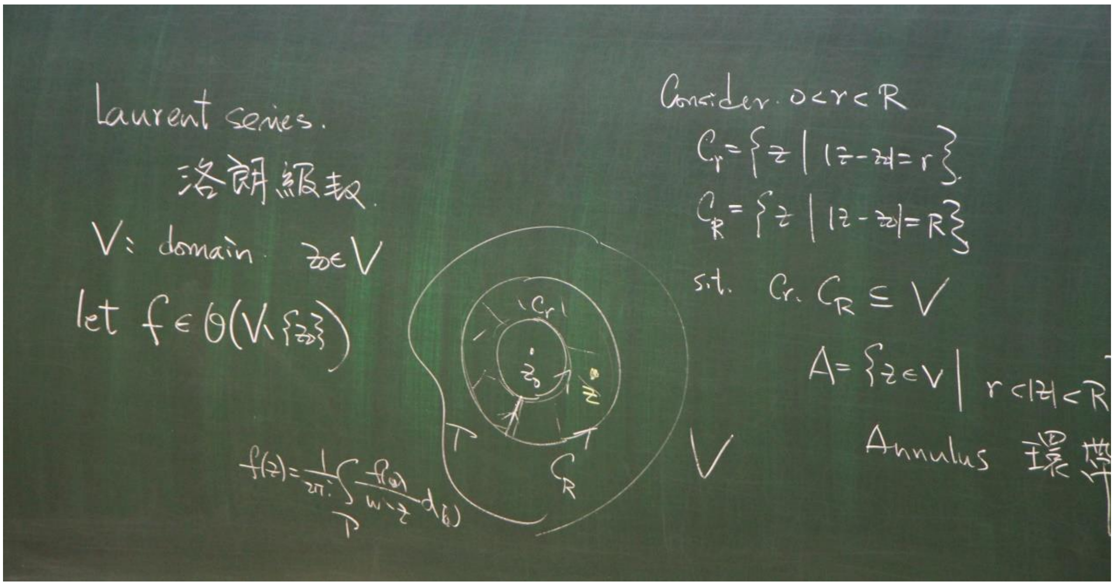
【10920 程守慶教授複變數函數論 / 第 11 堂版書】





series.

朗級數.

ain $z_0 \in V$

$\mathcal{O}(V \setminus \{z_0\})$

Consider $0<r<R$

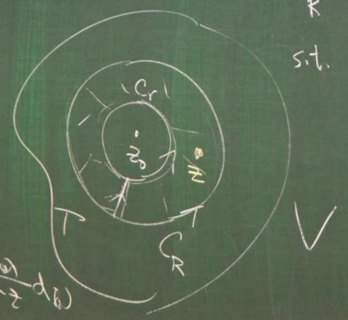
$C_r=\{z \mid |z-z_0|=r\}$

$C_R=\{z \mid |z-z_0|=R\}$

s.t. $C_r, C_R \subseteq V$

$A=\{z\in V \mid r<|z|<R\} \subseteq V$

Annulus 環帶



$$f(z)=\frac{1}{2\pi i}\int_{\gamma}\frac{f(w)}{w-z}\,dw$$

$$f(z)=\frac{-1}{2\pi i}\int_{C_r}\frac{f(w)}{w-z}\,dw+\frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w-z}\,dw$$

$$\frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w-z}\,dw=\frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w-z_0-(z-z_0)}\,dw$$

$$=\frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w-z_0}\cdot\frac{1}{1-\frac{z-z_0}{w-z_0}}\,dw$$

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$$=\frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w-z_0}\cdot\frac{1}{1-\frac{z-z_0}{w-z_0}}\,dw$$

$$=\frac{1}{2\pi i}\int_{C_R} f(w)\sum_{k=0}^{\infty}\frac{(z-z_0)^k}{(w-z_0)^{k+1}}\,dw$$

$$=\sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{(w-z_0)^{k+1}}\,dw\right)(z-z_0)^k$$

$$\frac{-1}{2\pi i}\int_{C_r}\frac{f(w)}{w-z}\,dw$$

$$= \frac{-1}{2\pi i}\int_{C_r}\frac{f(w)}{w-z_0-(z-z_0)}\,dw$$

$$= \frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{z-z_0}\,\frac{1}{1-\frac{w-z_0}{z-z_0}}\,dw$$

$$= \frac{1}{2\pi i}\int_{C_r} f(w)\sum_{k=0}^{\infty}\frac{(w-z_0)^k}{(z-z_0)^{k+1}}\,dw$$

$$= \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_r} f(w)(w-z_0)^k\,dw\right)\frac{1}{(z-z_0)^{k+1}}$$

$$\frac{(w)}{z}\,dw + \frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w-z}\,dw \;//\; \sum_{-\infty}^{\infty} a_k(z-z_0)^k$$

$$\boxed{a_k=\frac{1}{2\pi i}\int_{C}\frac{f(w)}{(w-z_0)^{k+1}}\,dw}$$

$$\frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w-z_0-(z-z_0)}\,dw$$

$$= \frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w-z_0}\cdot\frac{1}{1-\frac{z-z_0}{w-z_0}}\,dw$$

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$$= \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{(w-z_0)^{k+1}}\,dw\right)(z-z_0)^k$$

$$dw + \frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w-z}\,dw \;//\; \sum_{-\infty}^{\infty} a_n(z-z_0)^n$$

$$\boxed{a_n=\frac{1}{2\pi i}\int_{C}\frac{f(w)}{(w-z_0)^{n+1}}\,dw}$$

$$\frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w-z_0-(z-z_0)}\,dw$$

$$= \frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w-z_0}\cdot\frac{1}{1-\frac{z-z_0}{w-z_0}}\,dw$$

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$$= \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{(w-z_0)^{k+1}}\,dw\right)(z-z_0)^k$$

$$= \frac{1}{2\pi i}\int_{C_r} f(w) \sum_{k=0}^{\infty} \frac{(w-z_0)^k}{(z-z_0)^{k+1}}\, dw$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i}\int_{C_r} f(w)(w-z_0)^k\, dw \right) \frac{1}{(z-z_0)^{k+1}}$$

$$\frac{1}{2\pi i}\int_{C_r} f(w)(w-z_0)^{-n-1}\, dw$$

$$\|$$

$$\frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{(w-z_0)^{n+1}}\, dw$$

$\underline{\underline{n}} = -k-1 \qquad k = -n-1$

$$\boxed{\sum_{-\infty}^{\infty} a_n (z-z_0)^n} \qquad \boxed{a_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}}\, dw}$$

$$\| $$

$I_1 + I_2$.

I_1: $n \geq 0$. holomorphic part

I_2: $n < 0$, principal part.

① $a_n = 0$, $\forall\, n < 0$

$$\boxed{\sum_{-\infty}^{\infty} a_n (z-z_0)^n}$$

$$= I_1 + I_2 .$$

$$\boxed{a_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}}\, dw}$$

I_1 : $n \geq 0$. holomorphic part

I_2 : $n < 0$, principal part.

① $a_n = 0$, $\forall n < 0$ removable

② $a_n = 0$, when $n<0$, $|n| > n_0 \in \mathbb{N}$ some n_0.
$a_{-n_0} \neq 0$ pole

③ $a_n \neq 0$

$$\boxed{a_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}}\, dw}$$

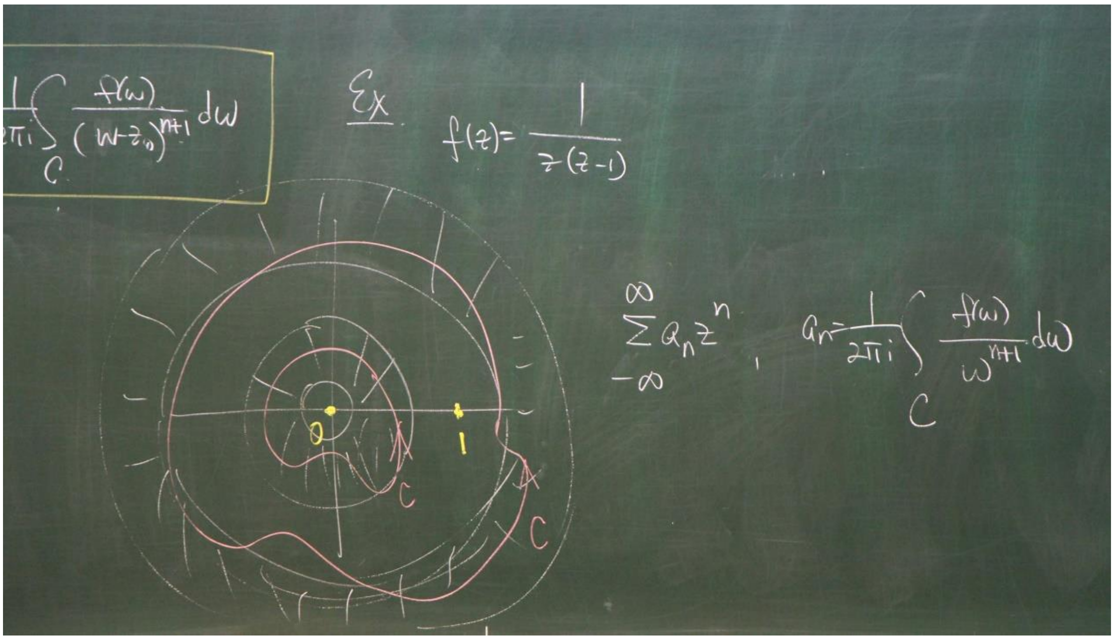
I_1 : $n \geq 0$. holomorphic part

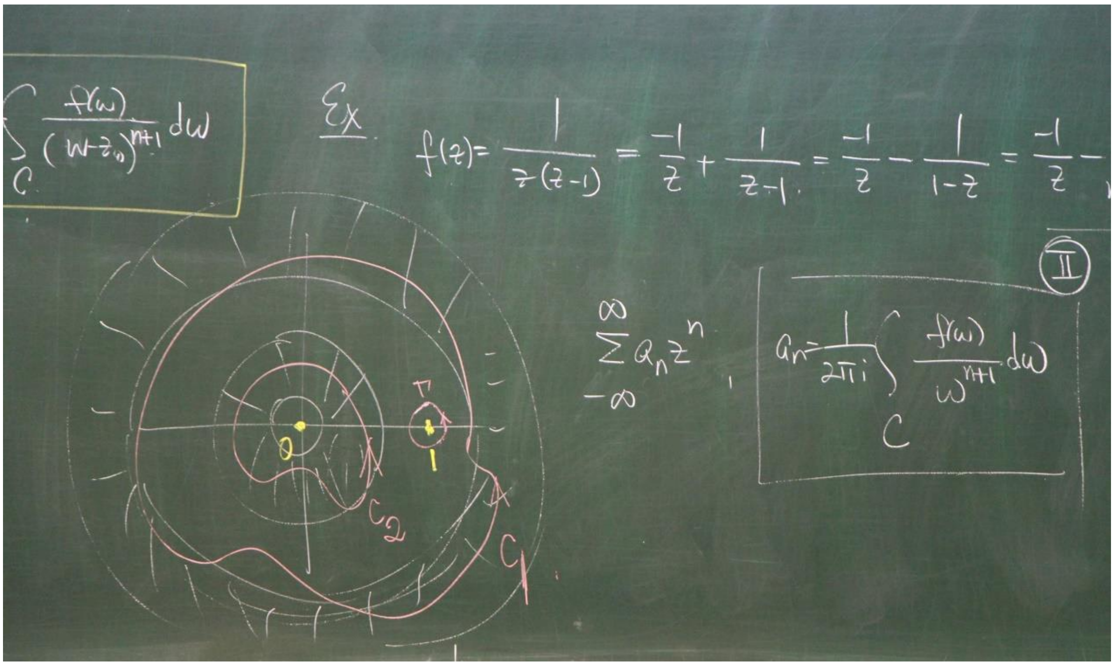
I_2 : $n < 0$, principal part.

$=0$, $\forall n < 0$ removable

$=0$, when $n<0$, $|n| > n_0 \in \mathbb{N}$ some n_0.
$a_{-n_0} \neq 0$ pole

③ $a_n \neq 0$ for infinitely many $n < 0$.
essential


$\frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}}\,dw$
Ex
$f(z) = \frac{1}{z(z-1)}$
$\sum_{-\infty}^{\infty} a_n z^n$,
$a_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{w^{n+1}}\,dw$
0
1
C
C


$\int_C \frac{f(w)}{(w-z_0)^{n+1}}\,dw$
Ex
$f(z) = \frac{1}{z(z-1)} = \frac{-1}{z} + \frac{1}{z-1} = \frac{-1}{z} - \frac{1}{1-z} = \frac{-1}{z} -$
II
$\sum_{-\infty}^{\infty} a_n z^n$,
$a_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{w^{n+1}}\,dw$
0
1
γ
C_2
C_1

$$\frac{1}{z(z-1)} = \frac{-1}{z} + \frac{1}{z-1} = \frac{-1}{z} - \frac{1}{1-z} = \frac{-1}{z} - \sum_{n=0}^{\infty} z^n = \sum_{-\infty}^{\infty} a_n z^n \qquad \begin{array}{l} a_n = 0, \ n \le -2 \\ a_{-1} = -1 \\ a_n = -1 \ \ n \ge 0 \end{array}$$

$$\sum_{-\infty}^{\infty} a_n z^n, \qquad \boxed{a_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{w^{n+1}}\,dw}$$

II

$$\frac{-1}{z} + \frac{1}{z-1} = \frac{-1}{z} + \frac{1}{z}\,\frac{1}{1-\frac{1}{z}}$$

$$= \frac{-1}{z} + \frac{1}{z}\sum_{k=0}^{\infty} \frac{1}{z^k}$$

$$= \frac{-1}{z} + \sum_{k=0}^{\infty} \frac{1}{z^{k+1}}$$

$$= \sum_{k=1}^{\infty} \frac{1}{z^{k+1}} = \sum_{-\infty}^{\infty} a_n z^n$$

$n \le -2$

$n \ge 0$) $n \ge -1$

$$= \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w^{n+1}}\,dw$$

$$\frac{1}{2\pi i}\int_{T} \frac{1}{w^{n+2}(w-1)}\,dw = 1 \quad \forall n \in \mathbb{Z}$$

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w^{n+1}}\,dw = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w^{n+1}}\,dw + \boxed{\frac{1}{2\pi i}\int_{T} \frac{f(w)}{w^{n+1}}\,dw}$$

$a_n = 0, \ n \ge -1$

$a_n = 1$ o.w.

Thm (Argument Principle)

幅角原理.

meromorphic functions

let D be a domain, $f \in \mathcal{M}(D)$

γ: simple closed curve in D. s.t.

the region bounded by γ is contained in D

Assume there is no zeros or poles of f sitting in γ.

Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}$$

N=

P

meromorphic functions

↓

$\in \mathcal{M}(D)$

D. s.t.

γ is contained in D

or poles of f sitting in γ.

Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = N - P.$$

N: number of zeros surrounded by γ (with multiplicities)

P: " " poles " "

$f(z_0) = 0$.

$f(z) = (z-z_0)^m \varphi(z)$

$\boxed{\underline{\underline{\text{Pole}}} \quad f(z) = (z-z_0)^{-m}\varphi(z) \quad m \in \mathbb{N}}$

$$\frac{1}{2\pi i}\int_{C_\varepsilon} \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_{C_\varepsilon} \frac{m(z-z_0)^{m-1}\varphi(z) + (z-z_0)^m\varphi'(z)}{(z-z_0)^m\varphi(z)}\,dz$$

$\frac{1}{2\pi}$

$$= \frac{1}{2\pi i}\int_{C_\varepsilon} \frac{m}{z-z_0}\,dz + \frac{1}{2\pi i}\int_{C_\varepsilon} \cancel{\frac{\varphi'(z)}{\varphi(z)}}\,dz \quad \to 0$$

$= m$

$=$

by γ

le $\quad f(z) = (z-z_0)^{-m}\varphi(z) \quad m \in \mathbb{N}$

$$\frac{m(z-z_0)^{m-1}\varphi(z) + (z-z_0)^{m}\varphi'(z)}{(z-z_0)^{m}\varphi(z)}\, dz$$

$$dz + \cancelto{0}{\frac{1}{2\pi i}\int_{C_\varepsilon} \frac{\varphi'(z)}{\varphi(z)}\, dz}$$

$$\frac{1}{2\pi i}\int_{C_\varepsilon} \frac{-m(z-z_0)^{-m-1}\varphi(z) + (z-z_0)^{-m}\varphi'(z)}{(z-z_0)^{-m}\varphi(z)}\, dz$$

$$= \frac{1}{2\pi i}\int_{C_\varepsilon} \frac{-m}{z-z_0}\, dz + \cancelto{0}{\frac{1}{2\pi i}\int_{C_\varepsilon} \frac{\varphi'(z)}{\varphi(z)}\, dz}$$

$$= -m$$

$$+ (z-z_0)^{-m}\varphi'(z) \quad dz$$

$\varphi(z)$

$$dz + \cancelto{0}{\frac{1}{2\pi i}\int_{C_\varepsilon} \frac{\varphi'(z)}{\varphi(z)}\, dz}$$

Thm

$f \in \mathcal{O}(D)$

Assume f has no zeros on γ

Then

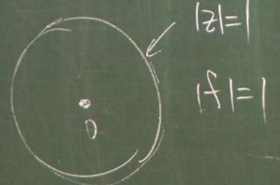
$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = N.$$

eros on γ

N.

Ex. $f(z) = z$.

U = open unit disc



take $0<\varepsilon<1$ $\quad \varepsilon < |f|$ on $|z|=1$

Set $g(z) = f(z)+\varepsilon = z+\varepsilon$.

$Z_g = \{-\varepsilon\}$

Thm (Rouché)

D: domain.

$\gamma \subseteq D$. simple closed curve

The region Ω bounded by γ is contained in D

let $f \in \mathcal{O}(D)$. $g \in \mathcal{O}(D)$.

s.t. $|g(z)| < |f(z)|$. $z \in \gamma$

Then. $N_f = N_{f+g}$. $\qquad N_f$ = number of zeros of f bounded by γ

pf.

$$N_{f+g} = \frac{1}{2\pi i}\int_\gamma \frac{f'+g'}{f+g}dz = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}dz = N_f$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f'+g'}{f+g}dz$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f'(1+\frac{g}{f})+g'}{f(1+\frac{g}{f})}$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}dz + \frac{1}{2\pi i}\int_\gamma$$

ed in D

$\Rightarrow$ f has no zeros on γ

of zeros of f bounded by γ

pf.

$$N_{f+g} = \frac{1}{2\pi i}\int_\gamma \frac{f'+g'}{f+g}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = N_f$$

...d in D

$\Longrightarrow$ f has no zeros on γ

... of zeros of f bounded by γ

$$\frac{1}{2\pi i}\int_\gamma \frac{f'+g'}{f+g}\,dz$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f'(1+\frac{g}{f})+g' \ldots}{f(1+\frac{g}{f})}$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz + \frac{1}{2\pi i}\int_\gamma \ldots$$

$$\ldots \frac{f'}{f}\,dz = N_f$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f'+g'}{f+g}\,dz$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f'(1+\frac{g}{f})+g'-\frac{f'}{f}g}{f(1+\frac{g}{f})}\,dz$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz + \frac{1}{2\pi i}\int_\gamma \frac{\frac{g'f-f'g}{f}}{f(1+\frac{g}{f})}\,dz$$

$$= N_f + \frac{1}{2\pi i}\int_\gamma \frac{(\frac{g}{f})'}{1+\frac{g}{f}}\,dz = N_f + \frac{1}{2\pi i}\int_\gamma \left(\frac{g}{f}\right)' \sum_{k=0}^{\infty}\left(-\frac{g}{f}\right)^k dz$$

$$\left|\frac{g}{f}\right| < 1 \text{ on } \gamma$$

$$= N_f + \sum_{k=0}^{\infty}(-1)^k \cdot \frac{1}{2\pi i}\int_\gamma \left(\frac{g}{f}\right)'\left(\frac{g}{f}\right)^k dz$$

$$= N_f + \sum_{k=0}^{\infty}(-1)^k \cdot \frac{1}{2\pi i}\,\frac{1}{k+1}\left(\frac{g}{f}\right)^{k+1}\Bigg|_{\text{initial point}}^{\text{end point}} = N_f.$$

$$= N_f + \frac{1}{2\pi i}\int_{\gamma} \frac{\left(\frac{g}{f}\right)'}{1+\frac{g}{f}}\, dz = N_f + \frac{1}{2\pi i}\int_{\gamma} \left(\frac{g}{f}\right)' \sum_{k=0}^{\infty} \left(-\frac{g}{f}\right)^k dz$$

$\left|\frac{g}{f}\right| < 1$ on γ

$$= N_f + \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{2\pi i}\int_{\gamma} \left(\frac{g}{f}\right)' \left(\frac{g}{f}\right)^k dz$$

$$= N_f + \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{2\pi i}\, \frac{1}{k+1}\left(\frac{g}{f}\right)^{k+1} \Bigg|_{\text{initial point}}^{\text{end point}} = N_f .$$

$-dz$

Thm (Hurwitz)

$f_n \in \mathcal{O}(D)$. f_n has no zero on D for all n

and $f_n \xrightarrow{\text{u.c.c.}} f$

then either f has no zero on D

or $f(z) \equiv 0$ on D

$f_n(z) = \frac{e^z}{n}$ on $\mathbb{C} \xrightarrow{\text{u.c.c.}} 0$.